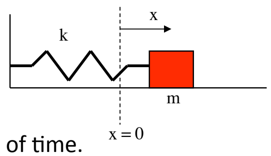
Problem 15.13

A particle moves through equilibrium into the *+x* side of the coordinate system at *t = 0 seconds.* Its amplitude is .02 m and its frequency is 1.5 Hz.



a.) Characterize the motion's *position* as a function of time.

Remember that,

$$x(t) = A\cos(\omega t + \phi)$$
$$= A\cos((2\pi\nu)t + \phi)$$

we can immediately write:

$$x(t) = (2.00\text{x}10^{-2}\ \text{m})\cos((2\pi(1.50\ \text{Hz}))t + \phi)$$
$$= (2.00\text{x}10^{-2}\ \text{m})\cos(3\pi t + \phi)$$

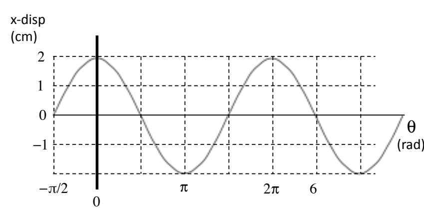
To complete this, all we need is the phase shift.

In fact, knowing what a cosine wave looks like, it should be easy to "eyeball" an answer here. There is a formal way of doing this, though, which I will outline.

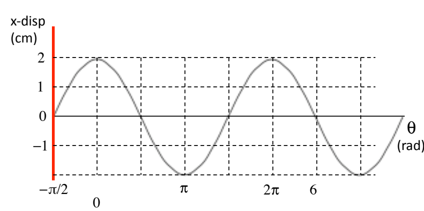
1.)

the Graphical approach:

1.) Start by sketching a sine or cosine wave, depending upon whether you want to characterize the position as a sine or cosine (we've decided to do it as a cosine).



2.) Find on the graph where you want the body to be at *t = 0*. In this case, we want it going through *x = 0* moving into the positive region. Once found, move your axis to that point (see new sketch).



3.) The phase shift will be the angular displacement required to get the axis into the right spot. In this case, that will be $-\pi/2$ radians so that

$$x(t) = (2.00\text{x}10^{-2}\ \text{m})\cos\left(3\pi t - \pi/2\right)$$

2.)

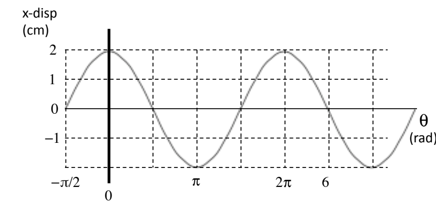
OR, the Mathematical approach (this is more sure-fire for complex problems like 15.22):

1.) You know where the body is and what it is doing at one point in time, which is to say, at *t = 0.* Use that information in the "x = . . . " expression to determine the one unknown in that expression, the *phase shift* ϕ. That is:

At *t = 0,* we know the body is passing through the origin so its *x-coordinate* will be zero, and we can write $x(t) = (2.00\text{x}10^{-2}\ \text{m})\cos(3\pi t + \phi)$ as:

$$x(t=0) = (2.00\text{x}10^{-2}\ \text{m})\cos(3\pi(0) + \phi)$$
$$\Rightarrow\quad 0 = (2.00\text{x}10^{-2}\ \text{m})\cos(\phi)$$
$$\Rightarrow\quad \cos(\phi) = 0$$
$$\Rightarrow\quad \phi = -\pi/2,\ +\pi/2,\ \text{etc.}$$

The trick is to decide which of the various angles will fit our situation. That is where a good sketch will help. But how to interpret it?



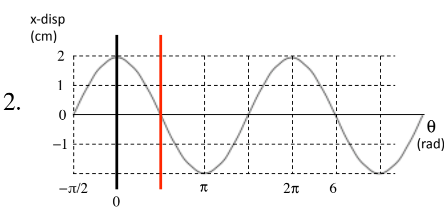
3.)

Again, the solution is to look and use your head.

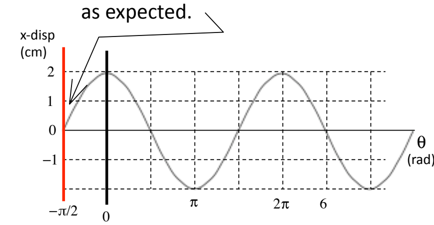
Let's assume the solution was $\phi = \pi/2$. Where would that put our axis?

In fact, it would slide it to the red position shown to the right. Does this satisfy our problem. If that was the correct position for the axis, would the body be moving through equilibrium at *t = 0*? (Yes, it would). And would it be proceeding into the *+x* region? (No, it wouldn't—the *+x* region is *above* the horizontal axis—in our graph, the body would being moving into the *−x* region.) So this isn't the shift we need.

Looking at the second graph, you can see that the appropriate shift is to the left by $\phi = -\pi/2$ radians.



After passing through the new origin, the body is moving into the +x region as expected.



Solution:

$$x(t) = (2.00\text{x}10^{-2}\ \text{m})\cos\left(3\pi t - \pi/2\right)$$

4.)

b.) What is the particle's *maximum speed*?

In general:

$$
\begin{aligned}
v &= \frac{dx}{dt} \\
&= \frac{d\left[A\cos(\omega t+\phi)\right]}{dt} \\
&= -\omega\left(A\sin(\omega t+\phi)\right)
\end{aligned}
$$

The maximum velocity will happen when the sin is as large as it will ever be, which is when it equals "1," so the magnitude of the maximum velocity will simply be ωA . Using that for our situation yields:

$$
\begin{aligned}
v_{max} &= \omega A \\
&= (3\pi \text{ rad/s})(2.00\text{x}10^{-2} \text{ m}) \\
&= 18.8 \text{ m/s}
\end{aligned}
$$

Notice that the units don't match up—we seem to have lost a "radians" in the solution. This is one of those instances when we have to acknowledge and accept that "radians" is not really a unit and, hence, can be ignored.

c.) Where is the *speed* maximum and when does this happen?

It will be maximum after the force has done all the accelerating that it can, which happens at equilibrium (that is where the force is zero). Another way to look at it is through common sense. Where does the body *look* like it's moving the fastest? Answer: When it passes through equilibrium.

As for when this happens, for this situation the body passes through equilibrium at *t = 0,* then again after half a *period*. As the period is the inverse of the *frequency*, and the *frequency* is 1.50 Hz, the *period* will be 2/3 seconds and half that will be 1/3 second.

d.) What is the particle's *maximum acceleration*?

In general:

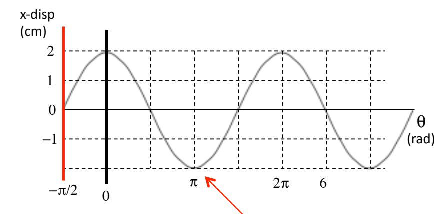
$$
\begin{aligned}
a &= \frac{dv}{dt} \\
&= \frac{d\left[-\omega A\sin(\omega t+\phi)\right]}{dt} \\
&= -\omega^2\left(A\cos(\omega t+\phi)\right)
\end{aligned}
$$

Again, the magnitude of the maximum acceleration will happen when the cosine is as large as it will ever be, which is when it equals "1," so the magnitude of the maximum velocity will simply be $\omega^2 A$. Using that for our situation yields:

$$
\begin{aligned}
a_{max} &= \omega^2 A \\
&= (3\pi \text{ rad/s})^2(2.00\text{x}10^{-2} \text{ m}) \\
&= 178 \text{ m/s}^2
\end{aligned}
$$

e.) Where is the *acceleration* maximum and positive, and when does this happen?

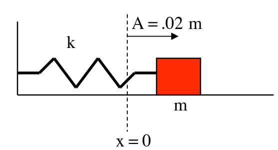
It will be maximum when the force is maximum *and positive*, which happens at the extremes of the vibratory motion when in the *negative region* beginning to move back in a *positive direction* toward equilibrium. In this case, that happens a three-quarters of the way through the *period* (see sketch), or at $t = \frac{3}{4}T = \frac{3}{4}(.667 \text{ s}) = .500 \text{ s}.$



starting back in + direction toward equilibrium ¾ way through cycle

f.) What is the total distance traveled between *t = 0* and *t = 1 second*.

This is a "use your head" problem. Think about what the motion looks like. In one period's worth of time, the body will travel through one full cycle. That is, it will travel out to its amplitude .02 meters from equilibrium, then travel another .02 meters back to equilibrium (that .04 meters so far), then do the same on the negative side of the origin for a total of .0800 meters per cycle.



You know one cycle takes 2/3 seconds as that was what we calculated for the *period*, so going to 1.0 seconds would be half as much again (2/3+1/3=1). In other words, the body will have traveled .08+.04 meters, or .120 meters during the time interval.